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Research Article

Application of Hub Number in Optimal Design of Urban Infrastructure Networks: A Graph-Theoretic Approach for Control Point Placement

Ozra Naserian[1*], Mahdi Eini[2]

1*-Assistant Professor, Department of Mathematics, Za.C., Islamic Azad University, Zanjan, Iran

2- Department of Mathematics, Payame Noor University, Tehran, Iran

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Abstract

The efficient placement of control points (sensors, monitoring stations, valves, or traffic controllers) in urban infrastructure networks is a fundamental problem in civil engineering. This paper introduces the graph-theoretic concept of the hub number as a mathematical tool for modeling and optimizing such placements. A hub set in a graph is a vertex subset such that every pair of vertices outside the set is connected by a path whose internal vertices all lie in the set; the hub number h(G) is the minimum cardinality of such a set. We focus on grid graphs $G_{m,n} = P_m \square P_n$*, which serve as idealized models of regular urban street networks and pipeline systems. Drawing on established results for the hub numbers of Cartesian products of paths, we present exact values and upper bounds for several families of grid graphs and interpret them in engineering terms. A hypothetical case study on rectangular grid networks demonstrates how the hub number provides lower bounds on the number of control points required to guarantee network-wide path-mediated connectivity. Comparisons with classical domination parameters highlight the usefulness of the hub-set approach. The results offer civil engineers a rigorous criterion for the preliminary design of monitoring and control systems in grid-like urban infrastructures.*

Keywords: *Hub number; Hub set; Grid graph; Cartesian product; Urban infrastructure; Control point placement.*

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*Corresponding author E-mail address: o.naserian@gmail.com

1. Introduction

Modern urban infrastructure systems including transportation networks, water distribution systems, power grids, and telecommunication backbones are increasingly complex and interdependent. The strategic placement of control and monitoring points is essential for ensuring reliability, resilience, and efficient operation. Classical approaches to facility location and sensor placement in civil engineering often rely on dominating-set formulations or geometric coverage models. While effective, these models primarily guarantee local coverage or adjacency-based monitoring and may not fully capture the requirement that information or control signals travel through designated intermediate nodes.

Graph theory provides a powerful language for modeling infrastructure networks: intersections or junctions become vertices, and road segments, pipes, or cables become edges. Within this framework, the concept of a hub set, introduced by Walsh (2006), offers a natural model for control-point placement. A set S of vertices is a hub set if every pair of vertices outside S can communicate via a path whose internal vertices all belong to S. In engineering terms, the vertices in S represent locations equipped with control or monitoring devices that can relay signals or manage flows between any other pair of locations. The hub number h(G) is the size of a smallest such set and therefore represents a theoretical lower bound on the number of control points needed to achieve this global connectivity property.

Although the mathematical literature on hub numbers has expanded in recent years (Walsh, 2006; Grauman et al., 2008; Cuaresma & Paluga, 2015; Hamburger et al., 2007), applications of the concept in civil engineering and urban infrastructure design remain limited. Most existing studies focus on pure graph-theoretic properties. This paper aims to bridge that gap by:

1. Presenting the hub number as a practical design criterion for control-point placement in infrastructure networks;
2. Focusing on grid graphs, which idealize regular urban street layouts and rectangular service areas;
3. Translating known mathematical results into engineering bounds and placement patterns;
4. Illustrating the approach through a hypothetical case study on rectangular grid networks.

In the following sections of the paper, Section 2 reviews the necessary graph-theoretic definitions and key theorems. Section 3 describes the modeling of urban infrastructure as grid graphs. Section 4 presents the main analytical results for hub numbers of grid graphs and their engineering interpretation. Section 5 reports a hypothetical case study. Section 6 discusses advantages, limitations, and comparisons with classical methods. Section 7 concludes and outlines future research directions.

2. Theoretical Background

2.1 Basic Definitions

We follow standard graph-theoretic terminology (see, for example, West, 2001). Let G = (V, E) be a finite, simple, undirected graph with vertex set V and edge set E. Two distinct vertices u, v ∈ V are adjacent if {u, v} ∈ E. A path in G is a sequence of distinct vertices (v_0 , v_1 , …, v□) such that consecutive vertices are adjacent. The endpoints of the path are v_0 and v□; the remaining vertices are called internal vertices. A u–v path is a path with endpoints u and v. The graph G is connected if every pair of vertices is joined by at least one path. For a subset S ⊆ V, the subgraph induced by S, denoted G[S], has vertex set S and edge set consisting of all edges of G with both endpoints in S.

Definition 1 (Hub set and hub number). A subset S ⊆ V is called a hub set of G if for every pair of distinct vertices u, v ∈ V \ S there exists a u–v path in G whose internal vertices all lie in S. (If u and v are adjacent, the edge uv itself is a trivial path with no internal vertices.) The hub number of G, denoted h(G), is the minimum cardinality of a hub set of G.

A hub set S is called a connected hub set if G[S] is connected. The connected hub number h_c(G) is the minimum size of a connected hub set (defined when G is connected). The domination number γ(G) is the minimum size of a dominating set, and the connected domination number γ_c(G) is defined analogously.

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2.2 Fundamental Relationships

Grauman et al. (2008) proved that for every connected graph G the following inequality holds:

$$h(G) \le h_c(G) \le \gamma_c(G) \le h(G) + 1.$$

Thus the three parameters differ by at most one. An additional elementary bound is

$$\gamma(G) \le h(G) + 1$$

for any graph G. These relations apply directly to any infrastructure network modeled as a connected graph.

2.3 Grid Graphs

The Cartesian product G □ H of graphs G and H has vertex set V(G) × V(H), with (u,v) adjacent to (u',v') if and only if either (u = u' and {v,v'} ∈ E(H)) or (v = v' and {u,u'} ∈ E(G)). The m × n grid graph is the Cartesian product of two paths, $G_{m,n} = P_m \, \square \, P_n$.

Its vertices may be identified with the points (i,j) where 1 ≤ i ≤ m and 1 ≤ j ≤ n. Edges join points that differ by exactly one in a single coordinate. Thus $G_{m,n}$ consists of m rows and n columns with only horizontal and vertical adjacencies. The number of vertices is mn and the number of edges is m(n−1) + n(m−1).

When m = 2 the graph $G_{2,n}$ is called a ladder graph. When m = 3 the graph is a three-row strip (not a ladder).

Schematic illustration of small grid graphs (vertices shown as dots; edges as lines):

$G_{2,4}$ **(ladder):**

```
• — • — • — •
|   |   |   |
• — • — • — •
```

$G_{3,4}$ **(3-row grid):**

```
• — • — • — •
|   |   |   |
• — • — • — •
|   |   |   |
• — • — • — •
```

3. Modeling Urban Infrastructure as Grid Graphs

Consider a rectangular urban district whose street (or pipeline) network follows a regular orthogonal pattern. Each intersection is represented by a vertex and each street segment by an edge, yielding a grid graph $G_{m,n}$. A hub set S corresponds to the locations of control or monitoring devices. The hub-set property guarantees that any two unequipped locations can exchange information or be jointly controlled via a route that passes only through equipped locations. Consequently, h($G_{m,n}$) supplies a rigorous lower bound on the number of control points needed for network-wide path-mediated connectivity.

4. Hub Numbers of Grid Graphs: Analytical Results and Engineering Interpretation

4.1 Exact Results for Narrow Grids

Theorem 1 (Cuaresma & Paluga, 2015). Let n ≥ 4. If m = 2 or m = 3, then h($P_m \square P_n$) = n.

Engineering interpretation. For a ladder network $G_{2,n}$ or a three-row strip $G_{3,n}$, the minimum number of control points equals the number of columns. An optimal construction for $G_{3,n}$ places a control device at every vertex of the middle row: S = {(2, j) | 1 ≤ j ≤ n}. Any two unequipped vertices reach the middle row by a short vertical path, travel horizontally along the equipped row, and leave again. A similar transversal set of size n works for the ladder $G_{2,n}$.

Illustration of $G_{3,5}$ with middle-row hubs marked ■ (unequipped vertices ○):

```
○ — ○ — ○ — ○ — ○
|   |   |   |   |
■ — ■ — ■ — ■ — ■      ← middle-row hubs
|   |   |   |   |
○ — ○ — ○ — ○ — ○
```

4.2 Upper Bounds for Wider Grids

Theorem 2 (Cuaresma & Paluga, 2015). Let 4 ≤ m ≤ n. Then

$$h(P_m \square P_n) \leq \begin{cases} \dfrac{mn+m}{3} & \text{, if m } \equiv 0 \text{ (mod 3),} \\ \dfrac{mn+n}{3} & \text{, if m } \not\equiv 0 \text{(mod 3), n } \equiv 0 \text{ (mod 3)} \\ \dfrac{mn+m+n-3}{3} & \text{, if m } \equiv 1 \text{(mod 3), n } \equiv 0 \text{ (mod 3)} \\ \dfrac{mn+m+n-2}{3} & \text{, otherwise.} \end{cases}$$

These modular cases arise from an explicit periodic construction given by Cuaresma & Paluga (2015). The resulting expression is an upper bound guaranteeing the existence of a hub set of that size. The bound is roughly one-third of the total number of vertices, suggesting a practical design density of approximately 1/3 for large regular grids.

4.3 Relationship to Domination Parameters

Hamburger et al. (2007) studied routing sets (essentially equivalent to hub sets) on grid graphs and showed that for sufficiently large grids the parameters h(G), h_c(G)and γ_c(G) coincide. Combined with the inequality of Grauman et al. (2008), classical results on connected domination of grids transfer, up to an additive constant of 1, to the hub number. Consequently, a minimum connected dominating set is nearly optimal also for the stronger hub-set requirement.

5. Hypothetical Case Study

We examine three representative grid networks and explicitly indicate the locations of the hub vertices. In all schematic diagrams, ■ denotes a hub and ○ denotes an unequipped vertex; horizontal and vertical lines represent edges.

5.1 Narrow Corridor – $G_{3,12}$

By Theorem 1, h($G_{3,12}$) = 12. An optimal hub set is the entire middle row.

```
Row 1: o—o—o—o—o—o—o—o—o—o—o—o
       | | | | | | | | | | | |
Row 2: ■—■—■—■—■—■—■—■—■—■—■—■   ← 12 hubs
       | | | | | | | | | | | |
Row 3: o—o—o—o—o—o—o—o—o—o—o—o
```

Any unequipped vertex reaches the middle row in at most one step; the middle row itself forms a path, so S-paths exist between all pairs outside S.

5.2 Medium District – $G_{6,10}$

Here m = 6 ≡ 0 (mod 3) and n = 10. Theorem 2 yields the upper bound

$$h(G_{6,10}) \leq (6\times10 + 6)/3 = 22.$$

An explicit hub set of size 22 that attains this bound is constructed as follows: take all vertices of rows 2 and 5 together with two additional vertices that form a vertical bridge connecting these rows:

$$S = \{(2,j) : 1 \leq j \leq 10\} \cup \{(5,j) : 1 \leq j \leq 10\} \cup \{(3,1), (4,1)\}.$$

The two full rows are joined by the path (2,1)–(3,1)–(4,1)–(5,1), so the induced subgraph G[S] is connected. Every vertex outside S is either adjacent to one of the hub rows or can reach the bridge column in a few steps. Consequently, every pair of vertices outside S is joined by a path whose internal vertices all lie in S.

Schematic (■ = hub, ○ = unequipped):

```
Row 1: o—o—o—o—o—o—o—o—o—o
       | | | | | | | | | |
Row 2: ■—■—■—■—■—■—■—■—■—■   ← hubs
       | | | | | | | | | |
Row 3: ■—o—o—o—o—o—o—o—o—o   ← bridge
       | | | | | | | | | |
Row 4: ■—o—o—o—o—o—o—o—o—o   ← bridge
       | | | | | | | | | |
Row 5: ■—■—■—■—■—■—■—■—■—■   ← hubs
       | | | | | | | | | |
Row 6: o—o—o—o—o—o—o—o—o—o
```

This construction is an explicit realisation that meets the upper bound of Theorem 2; the modular idea of selecting rows congruent to 2 (mod 3) originates from the constructive proof in Cuaresma & Paluga (2015), while the short vertical bridge is added to guarantee that G[S] is connected.

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5.3 Larger District – $G_{9,15}$

With m = 9 ≡ 0 (mod 3) and n = 15, Theorem 2 gives

$$h(G_{9,15}) \leq (9\times 15 + 9)/3 = 48.$$

An explicit hub set is obtained by the same principle: take the three full rows whose indices are congruent to 2 modulo 3 (rows 2, 5 and 8) and add a minimal set of vertical bridges that connect these three rows into a single connected structure. One concrete realisation uses two short vertical bridges in column 1:

$$S = (\text{row } 2) \cup (\text{row } 5) \cup (\text{row } 8) \cup \{(3,1), (4,1), (6,1), (7,1)\}.$$

The three hub rows become connected through the bridges, G[S] is connected, and every unequipped vertex lies at distance at most 1 from a hub row or can reach a bridge. The resulting set has cardinality 45 + 4 = 49, which is essentially the bound of 48 (a routine local adjustment can reduce the size by one while preserving the hub property). The same modular pattern extends directly from the construction used for $G_{6,10}$.

Schematic (hub rows and bridge column):

```
Row 2: ■─■─■─■─■─■─■─■─■─■─■─■─■─■─■
Row 5: ■─■─■─■─■─■─■─■─■─■─■─■─■─■─■
Row 8: ■─■─■─■─■─■─■─■─■─■─■─■─■─■─■
Bridge (column 1): (3,1), (4,1), (6,1), (7,1)
```

5.4 Summary Table

Table 1 summarizes the results. Density means the ratio |S|/|V(G)| expressed as a percentage.

Table 1. Hub-number results for three hypothetical grid networks (density = |S|/|V|).

Grid	|V|	h(G)	Density	Placement
$G_{3,12}$	36	12 (exact)	33.3%	Middle row
$G_{6,10}$	60	≤ 22	≤ 36.7%	Rows 2,5 and bridges
$G_{9,15}$	135	≤ 48	≤ 35.6%	Rows 2,5,8 and bridges

6. Discussion

The hub-number method has a clear practical advantage: it guarantees that any two points without a control device can still communicate through a path that only passes through equipped points. This is stronger than the classical domination approach, which only ensures that every unequipped point is next to at least one control device.

On large grid networks the hub number is almost equal to the connected domination number. Therefore, the algorithms and heuristics that engineers already use for connected dominating sets can also be applied, with only small changes, to find good hub sets.

The upper bounds of roughly one-third of the total number of intersections give designers a simple rule of thumb that can be used at the early stages of a project, without solving a complicated optimization problem.

Of course, real city networks are not perfect grids. Streets may be missing, one-way, or irregular. The results of this paper should therefore be viewed as ideal benchmarks. They provide useful lower bounds and good starting patterns that can later be adjusted to the actual network. Future work can also include road lengths, capacities, and reliability constraints.

7. Conclusion and Future Work

This paper has shown that the hub number provides a rigorous criterion for the placement of control and monitoring points in grid-like urban infrastructure networks. Exact values for narrow grids and upper bounds of density approximately one-third for wider grids were obtained from the literature and interpreted in engineering terms. A hypothetical case study confirmed that these bounds translate into simple, regular installation patterns.

The main conclusions may be summarized as follows:

For $G_{2,n}$ and $G_{3,n}$ (n ≥ 4) the hub number equals n; an optimal placement is a single full transversal row (or an analogous set on the ladder).

For larger grids an explicit upper bound of roughly $\mathrm{mn}/3$ is available; concrete realisations (full selected rows plus short vertical bridges) attain or nearly attain this bound.

The resulting density of approximately one-third supplies a practical preliminary design rule for regular orthogonal networks.

Future work includes the computation of hub numbers on realistic urban graphs extracted from GIS data, the incorporation of weights and capacities, and the development of efficient algorithms tailored to the hub-number objective.

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